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Representation type of the blocks of category $\mathcal{O}_S$ in types F_4 and G_2

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ABSTRACT

In this paper a complete classification of the representation type of the infinitesimal blocks of the parabolic category $\mathcal{O}_S$ for the complex simple Lie algebras of types F_4 and G_2 is given.

1. Introduction

Indecomposable modules of a finite-dimensional algebra provide a complete description of all the modules of the algebra. Consequently, classifying the indecomposable modules for a fixed finite-dimensional algebra is a central theme in the representation theory of such algebras. One of the first questions one can ask is, "How 'classifiable' are the indecomposable modules of a certain finite-dimensional algebra?" A finite-dimensional algebra will fall into one of three classes depending on the 'classifiability' of its indecomposable modules. If there are only finitely many isomorphism classes of indecomposable modules, then we say the algebra has *finite representation type*. If there are infinitely many isomorphism classes of indecomposable modules, then we say the algebra has *infinite representation type*. If the algebra has infinite representation type, it can be further classified as having *tame representation type* if, roughly speaking, these indecomposable modules can be parameterized in some way, and *wild representation type* otherwise (see [Dro,CB]).

Cline, Parshall, and Scott [CPS, Thm. 3.6] proved that a highest weight category with finitely many simple objects is equivalent to the category of finitely generated modules of some (finite-dimensional) quasi-hereditary algebra. Projective modules of quasi-hereditary algebras admit filtrations by certain

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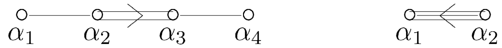

Fig. 1. Dynkin diagrams for F_4 (left) and G_2 (right).

standard modules. Consequently, at times it is possible to directly deduce the structures of the projective modules. Using this information, one can express the algebra as a quiver with relations from which one can potentially determine the representation type of the algebra.

If S is a subset of simple roots of a root system of a complex simple Lie algebra $\mathfrak{g}$ relative to a fixed maximal toral subalgebra, then category $\mathcal{O}_S$ is a generalization of the ordinary Berstein–Gelfand–Gelfand category $\mathcal{O}$. These are highest weight categories having a decomposition into infinitesimal blocks, with each block equivalent to the module category of some finite-dimensional quasi-hereditary algebra. One is justified in talking about the representation type of an infinitesimal block of $\mathcal{O}_S$ because of this underlying algebra.

If μ is an integral weight, then the infinitesimal block $\mathcal{O}_S^\mu$ contains all the simple modules with highest weight linked to μ via the dot action of the Weyl group. The singular root system of μ has simple roots J contained in the set of simple roots of the root system of $\mathfrak{g}$. Hence, the (singular) integral infinitesimal blocks of category $\mathcal{O}_S$ are determined by subsets J of the simple roots.

The classification of the representation type of the infinitesimal blocks of category $\mathcal{O}_S$ began with the independent work of Futorny, Nakano, and Pollack [FNP] and Brüstle, König, and Mazorchuk [BKM] in classifying the representation type of the blocks of ordinary category $\mathcal{O}$ (where $S = \emptyset$). Boe and Nakano [BN] later classified the representation type of all infinitesimal blocks of $\mathcal{O}_S$ with $S \cap J = \emptyset$.

In this paper, the representation type of all integral infinitesimal blocks of category $\mathcal{O}_S$ for any S will be determined in the case when $\mathfrak{g}$ is of type F_4 or G_2. A computer was employed to determine the representation type of the infinitesimal blocks. After tabulating the results, one can observe a natural partition of the set of sixteen (resp., four) subroot systems of F_4 (resp., G_2) into nine (resp., 3) equivalence classes. Furthermore, for both root systems, one can define a partial ordering on these equivalence classes which encapsulates the classification of the representation type of the infinitesimal blocks (see Theorems 5 and 6).

The paper is organized as follows. The necessary preliminaries and machinery are developed in Section 2. Then in Section 4 some criteria will be developed for determining the representation type of a given infinitesimal block. Section 5 is devoted to determining the representation type of all infinitesimal blocks of category $\mathcal{O}_S$ when $\mathfrak{g}$ is of type F_4 or G_2.

2. Preliminaries

2.1. Notation

Let $\mathfrak{g}$ be a simple Lie algebra over the field $\mathbb{C}$ of complex numbers with root system Φ of type F_4 or G_2, determined by a maximal toral subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. If $n \in \{2, 4\}$ is the rank of Φ, then let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be a set of simple roots for Φ, ordered as given in the Dynkin diagrams for F_4 and G_2 in Fig. 1. Let Φ^+ and Φ^- be (respectively) the set of positive roots and the set of negative roots with respect to Δ.

For each $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ be the α root space. If $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) is the sum of the positive (resp. negative) root spaces, then we have the Cartan decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, the Borel subalgebra defined by $\mathfrak{h}$ and Δ.

Denote the standard inner product on $\mathfrak{h}^*$ by $(-,-)$. For each $\alpha \in \Phi \subseteq \mathfrak{h}^*$, let s_α denote the reflection corresponding to α. For a simple root $\alpha_i \in \Delta$, we will write the corresponding simple reflection as $s_i := s_{\alpha_i}$. Let W denote the Weyl group generated by the simple reflections. The length of $w \in W$ will be denoted $l(w)$. The relation $w_1 \leqslant w_2$ denotes the Bruhat ordering on W. The Weyl group acts on $\mathfrak{h}^*$ via the dot action:

$$w \cdot \mu = w(\mu + \rho) - \rho \quad \text{for all } w \in W,\ \mu \in \mathfrak{h}^*,$$

where ρ is the half sum of the positive roots.

Write $\mathbb{Z}$ for the integers and $\mathbb{Z}_{\geqslant 0}$ for the non-negative integers. Denote the coroot of $\alpha \in \Phi$ by $\check{\alpha} := \frac{2\alpha}{(\alpha,\alpha)}$. Let $X = \{\mu \in \mathfrak{h}^* \mid (\mu, \check{\alpha}) \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$ denote the integral weight lattice, and let $X^+ = \{\mu \in X \mid (\mu, \check{\alpha}) \in \mathbb{Z}_{\geqslant 0} \text{ for all } \alpha \in \Phi^+\}$ denote the set of dominant integral weights.

2.2. The category $\mathcal{O}_S$

Fix $S \subseteq \Delta$, viewed where appropriate as a subset of $\{1, \ldots, n\}$ via the fixed ordering on simple roots. Then S determines a standard parabolic subalgebra $\mathfrak{p}_S = \mathfrak{m}_S \oplus \mathfrak{u}_S^+ \supseteq \mathfrak{b}$ of $\mathfrak{g}$ (see [RC]). The Lie subalgebras $\mathfrak{m}_S$ and $\mathfrak{u}_S^+$ are called, respectively, the Levi factor and the nilradical. The subset

$$\Phi_S = \Phi \cap \sum_{\alpha \in S} \mathbb{Z}\alpha$$

of Φ is the root system of $\mathfrak{m}_S$ with simple roots S and positive roots $\Phi_S^+ := \Phi^+ \cap \Phi_S$. Denote the Weyl group of Φ_S by W_S, viewed as a subgroup of W. Let w_S denote the longest element of W_S.

Let $\mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$. We will work with $\mathcal{U}(\mathfrak{g})$-modules in the parabolic category $\mathcal{O}_S$, defined as follows (see [RC]).

Definition 1. Let $\mathcal{O}_S$ be the full subcategory of the category of $\mathcal{U}(\mathfrak{g})$-modules consisting of modules V which satisfy the following conditions:

(i) V is a finitely generated $\mathcal{U}(\mathfrak{g})$-module.
(ii) As a $\mathcal{U}(\mathfrak{m}_S)$-module, V is the direct sum of finite-dimensional $\mathcal{U}(\mathfrak{m}_S)$-modules.
(iii) If $v \in V$, then $\dim_{\mathbb{C}} \mathcal{U}(\mathfrak{u}_S^+)v < \infty$.

Define $X_S^+ = \{\mu \in \mathfrak{h}^* \mid (\mu, \check{\alpha}) \in \mathbb{Z}_{\geqslant 0} \text{ for all } \alpha \in \Phi_S^+\}$. The key objects in category $\mathcal{O}_S$ are the parabolic Verma modules, which are constructed as follows. Start with a finite-dimensional simple $\mathfrak{m}_S$-module $F(\mu)$ with highest weight $\mu \in X_S^+$ ($F(\mu)$ is finite-dimensional if and only if $\mu \in X_S^+$). Extend $F(\mu)$ to a $\mathfrak{p}_S$-module by letting $\mathfrak{u}_S^+$ act by zero. The induced module

$$V(\mu) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}_S)} F(\mu)$$

is a *parabolic Verma module* (or PVM for short). These are also called *generalized Verma modules* in the literature.

$V(\mu)$ has the following properties. First, it is a highest weight module for $\mathfrak{g}$ with highest weight μ. Hence, it is a quotient of the ordinary Verma module $M(\mu)$ having highest weight μ. Also, $V(\mu)$ is an object with finite length in category $\mathcal{O}_S$. Furthermore, $V(\mu)$ has a unique maximal submodule and hence a unique simple quotient module, which we denote by $L(\mu)$; $L(\mu)$ is also the unique simple quotient of $M(\mu)$. Each simple module in $\mathcal{O}_S$ is isomorphic to some $L(\mu)$.

Every module in $\mathcal{O}_S$ has a projective cover, and so there is a one-to-one correspondence between the simple modules and the projective indecomposable modules in $\mathcal{O}_S$ (see [RC]). Let $P(\mu)$ be the projective cover of $L(\mu)$ in $\mathcal{O}_S$ for each $\mu \in X_S^+$.

Every projective module P in $\mathcal{O}_S$ has a *parabolic Verma composition series*:

$$P = P_0 \supseteq P_1 \supseteq \cdots \supseteq P_{r-1} \supseteq P_r = 0$$

such that $P_{i-1}/P_i = V(\nu_i)$ for some $\nu_i \in X_S^+$ ($1 \leqslant i \leqslant r$) (see [RC]). We have the following reciprocity law:

$$\big[P(\mu) : V(\nu)\big] = \big[V(\nu) : L(\mu)\big] \quad \text{for all } \mu, \nu \in X_S^+.$$

2.3. Infinitesimal blocks of $\mathcal{O}_S$

Let Z be the center of $\mathcal{U}(\mathfrak{g})$ and denote the set of algebra homomorphisms $Z \to \mathbb{C}$ by $Z^\sharp$. We say that $\chi \in Z^\sharp$ is the *infinitesimal character* of some nonzero $V \in \mathcal{O}_S$ if $zv = \chi(z)v$ for all $z \in Z$ and all $v \in V$. For each $\chi \in Z^\sharp$, let $\mathcal{O}_S^\chi$ be the full subcategory of $\mathcal{O}_S$ consisting of modules $V \in \mathcal{O}_S$ such that for all $z \in Z$, each $v \in V$ is annihilated by some power of $z - \chi(z)$. We have the decomposition

$$\mathcal{O}_S = \bigoplus_{\chi \in Z^\sharp} \mathcal{O}_S^\chi$$

of the category $\mathcal{O}_S$ in which each module in $\mathcal{O}_S$ decomposes into a direct sum of modules with each summand belonging to one of the subcategories $\mathcal{O}_S^\chi$. We call $\mathcal{O}_S^\chi$ an *infinitesimal block* of category $\mathcal{O}_S$, corresponding to the infinitesimal character χ.

For each $\mu \in \mathfrak{h}^*$, the ordinary Verma module $M(\mu)$ (and any quotient thereof, such as $V(\mu)$ or $L(\mu)$ if $\mu \in X_S^+$) has an infinitesimal character which we will denote by $\chi_\mu \in Z^\sharp$. Furthermore, if $\chi \in Z^\sharp$ is an infinitesimal character, then there exists $\mu \in \mathfrak{h}^*$ such that $\chi = \chi_\mu$. If $\chi = \chi_\mu$, we can write $\mathcal{O}_S^\mu = \mathcal{O}_S^{\chi_\mu} = \mathcal{O}(\mathfrak{g}, S, \mu)$ for $\mathcal{O}_S^\chi$. The Harish-Chandra linkage principle yields

$$\chi_\mu = \chi_\nu \quad \Longleftrightarrow \quad \nu \in W \cdot \mu.$$

Thus, $V(\nu)$ (resp. $L(\nu)$, $P(\nu)$) is in $\mathcal{O}_S^\mu$ if and only if $\nu = w_S w \cdot \mu$ for some $w \in W$. Since PVM's are constructed from the finite-dimensional $\mathfrak{m}_S$-modules with highest weights in X_S^+, the set of PVM's in $\mathcal{O}_S^\mu$ is $\{V(w_S w \cdot \mu) \mid w_S w \cdot \mu \in X_S^+\}$. Consequently, the PVM's (as well as the simple modules and projective indecomposable modules) in $\mathcal{O}_S^\mu$ are parameterized by $\{w \in W \mid w_S w \cdot \mu \in X_S^+\}$.

Assume from now on that μ is an integral weight and $\mu + \rho$ is antidominant; i.e., $(\mu + \rho, \check{\alpha}) \in \mathbb{Z}_{\leqslant 0}$ for all $\alpha \in \Delta$; (if it is not antidominant, we can replace it by a W-translate, so we are justified in making this assumption). Let

$$\Phi_\mu = \bigl\{\alpha \in \Phi \bigm| (\mu + \rho, \check{\alpha}) = 0\bigr\}.$$

If $\Phi_\mu = \emptyset$, then $\mu + \rho$ is a *regular weight*. If $\mu + \rho$ and $\nu + \rho$ are both regular weights, then $\mathcal{O}_S^\mu$ is equivalent to $\mathcal{O}_S^\nu$ by the Jantzen–Zuckerman translation principle. If $\mu + \rho$ is a regular weight, then $\{w \in W \mid w_S w \cdot \mu \in X_S^+\}$ is the set

$$\begin{aligned} {}^S W &= \bigl\{w \in W \bigm| l(s_\alpha w) = l(w) + 1 \text{ for all } \alpha \in S\bigr\} \\ &= \bigl\{w \in W \bigm| w^{-1}\bigl(\Phi_S^+\bigr) \subseteq \Phi^+\bigr\} \end{aligned}$$

which is the set of smallest length representatives for the right cosets of W_S in W.

Now, if $\mu \in \mathfrak{h}^*$ is such that $\Phi_\mu \neq \emptyset$, then Φ_μ is a subroot system of Φ, and in this case $\mu + \rho$ is a *singular weight*. Suppose $\alpha \in \Phi^+ \cap \Phi_\mu$. Then $(\mu + \rho, \check{\alpha}) = 0$ and also we can write $\alpha = \sum_{i=1}^n a_i \alpha_i$ for some $a_i \in \mathbb{Z}_{\geqslant 0}$. Since $\mu + \rho$ is antidominant, $(\mu + \rho, \check{\alpha}_i) \leqslant 0$ for each $1 \leqslant i \leqslant n$. Consequently, $0 = (\mu + \rho, \alpha) = \sum_{i=1}^n a_i(\mu + \rho, \alpha_i)$ implies that $(\mu + \rho, \alpha_i) = 0$ for any $i \in \{1, \ldots, n\}$ such that $a_i \neq 0$. Set

$$J = \bigl\{\alpha \in \Delta \bigm| (\mu + \rho, \alpha) = 0\bigr\}.$$

Then Φ_μ is the root system Φ_J which has simple roots J. Note that the Weyl group W_J of Φ_J is the stabilizer of $\mu + \rho$.

If $\mu + \rho$ is singular, then $w_S w \cdot \mu \in X_S^+$ if and only if $wW_J \subseteq {}^S W$. Since W_J stabilizes $\mu + \rho$, the set

$$^{S}W^{J} = \{ w \in {}^{S}W \mid w < ws_{\alpha} \in {}^{S}W \text{ for all } \alpha \in J \}$$

is the set of smallest length representatives for the left cosets wW_J contained in ^{S}W. Consequently, the set $^{S}W^{J}$ parameterizes the set of inequivalent irreducible modules in the infinitesimal block $\mathcal{O}_S^{\mu}$. That is, the set of simple modules in $\mathcal{O}_S^{\mu}$ is the set $\{L(w_S w \cdot \mu) \mid w \in {}^{S}W^{J}\}$ (see [BN]).

As with S, we will frequently view the set J as a subset of $\{1, \dots, n\}$. We will use the notation

$$\mathcal{O}_S^{\mu} = \mathcal{O}(\Phi, S, J) = \mathcal{O}(\mathfrak{g}, S, J)$$

when $\Phi_{\mu} = \Phi_J$.

3. Radical filtrations and the U_{α}-algorithm

3.1. Radical filtrations and extensions

The *radical* of a $\mathfrak{g}$-module V, denoted $\operatorname{rad} V$ is the smallest submodule of V such that $V/\operatorname{rad} V$ is semisimple. If V is a $\mathfrak{g}$-module, set $\operatorname{rad}^0 V = V$ and for each $i \geqslant 1$, set $\operatorname{rad}^i V = \operatorname{rad}(\operatorname{rad}^{i-1} V)$. We thus have the *radical filtration* of V:

$$V = \operatorname{rad}^0 V \supseteq \operatorname{rad}^1 V \supseteq \operatorname{rad}^2 V \supseteq \cdots.$$

If V is a finite length module (i.e., all chains of submodules in V have finite length), then for each $i \geqslant 0$, define $\operatorname{rad}_i V = \operatorname{rad}^i V / \operatorname{rad}^{i+1} V$, which is called the *ith radical layer* of V. Each PVM has a finite radical filtration.

Let A be a finite-dimensional algebra over $\mathbb{C}$. Then A is Morita equivalent to some basic algebra Λ. Let $L_1, \dots, L_r$ be a complete set of non-isomorphic simple Λ-modules with corresponding projective covers $P_1, \dots, P_r$. The $\operatorname{Ext}^1_{\Lambda}$-quiver $Q(\Lambda)$ of Λ has its vertices in one-to-one correspondence with the simple modules $\{L_i\}$, and the number of arrows from vertex i to vertex j is equal to $\dim_{\mathbb{C}} \operatorname{Ext}^1_{\Lambda}(L_i, L_j) = \dim_{\mathbb{C}} \operatorname{Hom}_{\Lambda}(P_j, \operatorname{rad}(P_i)) / \operatorname{Hom}_{\Lambda}(P_j, \operatorname{rad}^2(P_i))$. From a theorem of Gabriel [Gab2], the basic finite-dimensional algebra Λ is isomorphic to $\mathbb{C}Q(\Lambda)/I$ for some ideal I of the path algebra $\mathbb{C}Q(\Lambda)$; therefore, the category of A-modules is equivalent to the category of representations of some path algebra of a quiver with relations.

Every extension between two irreducible modules in $\mathcal{O}_S$ arises from an extension between them in layers 0 and 1 of the radical filtration of some PVM. In fact, if $\Phi_{\mu} = \Phi_J$ and $x, w \in {}^{S}W^{J}$ satisfies $x < w$, we have

$$\dim \operatorname{Ext}^1_{\mathcal{O}_S}\big(L(w_S w \cdot \mu), L(w_S x \cdot \mu)\big) = \big[\operatorname{rad}_1 V(w_S w \cdot \mu) : L(w_S x \cdot \mu)\big].$$

Furthermore,

$$\operatorname{Ext}^1_{\mathcal{O}_S}\big(L(w_S w \cdot \mu), L(w_S x \cdot \mu)\big) \cong \operatorname{Ext}^1_{\mathcal{O}_S}\big(L(w_S x \cdot \mu), L(w_S w \cdot \mu)\big)$$

(see [BN, Sec. 2.3]). In particular, one has that if $[\operatorname{rad}_1 V(w_S w \cdot \mu) : L(w_S x \cdot \mu)] \neq 0$, then the Ext^1-quiver for $\mathcal{O}_S$ has an arrow from $L(w_S w \cdot \mu)$ to $L(w_S x \cdot \mu)$, and an arrow from $L(w_S x \cdot \mu)$ to $L(w_S w \cdot \mu)$.

An infinitesimal block $\mathcal{O}_S^{\mu}$ is *semisimple* if and only if there are no non-split extensions between its simple modules. Because there are no self-extensions between simple modules in a highest weight category, an infinitesimal block with only one PVM (and hence only one simple module) is necessarily semisimple.

Let $\mathcal{S}$ be a set of simple modules in $\mathcal{O}_S^{\mu}$ corresponding to all the vertices in a single graph component of the Ext^1-quiver associated to $\mathcal{O}_S^{\mu}$. The full subcategory of $\mathcal{O}_S^{\mu}$ consisting of those modules whose composition factors are contained in $\mathcal{S}$ is called a *linkage class* of $\mathcal{O}_S^\mu$. It is apparent that $\mathcal{O}_S^\mu$ is semisimple if and only if $\mathrm{rad}_1 V = 0$ for all PVM's V in $\mathcal{O}_S^\mu$ if and only if each linkage class of $\mathcal{O}_S^\mu$ is composed of a single simple module.

3.2. The U_α-algorithm

The U_α algorithm is a tool used to compute radical filtrations of PVM's in an infinitesimal block $\mathcal{O}_S^\mu$ (see [Irv,Vog]).

First, let λ be a regular antidominant integral weight. Fix a simple reflection s_α for some $\alpha \in \Delta$. If one composes the translation functors 'onto' and 'out of' the α-wall, one gets an exact covariant functor θ_α on $\mathcal{O}_S$ called *translation through the α-wall*. For $w \in {}^SW$, $\theta_\alpha L(w_S w \cdot \lambda) = 0$ unless $w < ws_\alpha \in {}^SW$; in this case, $\theta_\alpha L(w_S w \cdot \lambda)$ has radical filtration layers

$$\begin{gathered} L(w_S w \cdot \lambda), \\ U_\alpha L(w_S w \cdot \lambda), \\ L(w_S w \cdot \lambda), \end{gathered}$$

where $U_\alpha L(w_S w \cdot \lambda)$ is a semisimple module defined as follows. Let

$$\mathcal{W} = \{x \in {}^SW \mid x > xs_\alpha \text{ or } xs_\alpha \notin {}^SW\}.$$

For $x, w \in {}^SW$ with $x < w$, let $\mu_S(x, w)$ be the coefficient of $q^{(l(w)-l(x)-1)/2}$ in the relative Kazhdan–Lusztig polynomial $P^S_{x,w}(q)$, called the relative Kazhdan–Lusztig μ-function (see [CC]). In fact, $\mu_S(x, w) = [\mathrm{rad}_1 V(w_S w \cdot \lambda) : L(w_S x \cdot \lambda)]$ (see [BN]). Now,

$$U_\alpha L(w_S w \cdot \lambda) = L(w_S w s_\alpha \cdot \lambda) \oplus \bigoplus_{x \in \mathcal{W}} \mu_S(x, w) L(w_S x \cdot \lambda).$$

One can start with $V(w_S e \cdot \lambda) = L(w_S e \cdot \lambda)$ and use the fact that if $w \in {}^SW$ with $w < ws_\alpha \in {}^SW$, then $\theta_\alpha V(w_S w \cdot \lambda)$ is a non-split extension of $V(w_S w s_\alpha \cdot \lambda)$ by $V(w_S w \cdot \lambda)$ to compute inductively the composition factors of each $V(w_S w \cdot \lambda)$.

Using a 'graded' version of the U_α-algorithm, one can compute not just the composition factors but also the radical filtrations of the PVM's (see [Bac,BGS,BN,Irv,Str]). Given a module M with filtration $\{M^i\}$, define σM to be the same module with filtration $(\sigma M)^i = M^{i-1}$. Suppose $w, ws_\alpha \in {}^SW$ with $w < ws_\alpha$ and that the radical filtration of $V(w_S w \cdot \lambda)$ is known. Compute the radical filtration $V = \mathrm{rad}^0 V \supseteq \mathrm{rad}^1 V \supseteq \mathrm{rad}^2 V \supseteq \cdots$ of $V := V(w_S w s_\alpha \cdot \lambda)$ as follows. First, the module $\theta_\alpha V(w_S w \cdot \lambda)$ has the following filtration. For each $i \geqslant 0$, let $L(w_S y \cdot \lambda)$ be a composition factor of $\mathrm{rad}_i V(w_S w \cdot \lambda)$ with $y, ys_\alpha \in {}^SW$ and $y < ys_\alpha$ (so that $\theta_\alpha L(w_S y \cdot \lambda) \neq 0$). If $j = 0, 1, 2$, then $\mathrm{rad}_j \theta_\alpha L(w_S y \cdot \lambda)$ occurs in the $(i + j)$th layer of $\theta_\alpha V(w_S w \cdot \lambda)$. There is a short exact sequence

$$0 \to \sigma V(w_S w s_\alpha \cdot \lambda) \to \theta_\alpha V(w_S w \cdot \lambda) \to V(w_S w \cdot \lambda) \to 0$$

of filtered modules. Hence, deleting the known radical filtration of $V(w_S w \cdot \lambda)$ from $\theta_\alpha V(w_S w \cdot \lambda)$ leaves the radical filtration of $V(w_S w s_\alpha \cdot \lambda)$ (with all layers shifted up in index).

Now suppose that μ is any antidominant integral weight and let $J \subseteq \Delta$ be the set of simple roots on which $\mu + \rho$ is singular. If $x, w \in {}^SW^J$, then

$$[\mathrm{rad}_i V(w_S w \cdot \mu) : L(w_S x \cdot \mu)] = [\mathrm{rad}_i V(w_S w \cdot \lambda) : L(w_S x \cdot \lambda)].$$

Consequently, the radical filtration of $V(w_S w \cdot \mu)$ is obtained from that of $V(w_S w \cdot \lambda)$ by ignoring all simple modules $L(w_S y \cdot \lambda)$ for $y \notin {}^SW^J$.

4. Representation type of infinitesimal blocks of category $\mathcal{O}_S$

In this section, we compile some criteria to determine the representation type of a given infinitesimal block of category $\mathcal{O}_S$. These criteria can be used whenever the structure of the PVM's in the infinitesimal block $\mathcal{O}_S^\mu$ is known; however, one criterion for wild representation type depends only on knowing something about the Bruhat order on ${}^S W^J$ (Proposition 4).

4.1. Triangular infinitesimal blocks

Suppose a linkage class of $\mathcal{O}(\mathfrak{g}, S, J)$ has m simple modules, labeled $L_1, \dots, L_m$. If the corresponding PVM's $V_1, \dots, V_m$ have radical filtration layers

$$\begin{array}{cccccc} V_1 & V_2 & V_3 & \cdots & V_{m-1} & V_m \\ \hline L_1 & L_2 & L_3 & \cdots & L_{m-1} & L_m \\ & L_1 & L_2 & \cdots & L_{m-2} & L_{m-1} \\ & & L_1 & \cdots & L_{m-3} & L_{m-2} \\ & & & \ddots & \vdots & \vdots \\ & & & & L_1 & L_2 \\ & & & & & L_1 \end{array} \tag{4.1.1}$$

then we say that the linkage class of $\mathcal{O}(\mathfrak{g}, S, J)$ is a *triangular linkage class of length m*. If $\mathcal{O}(\mathfrak{g}, S, J)$ has only one linkage class and it is triangular of length m, then we say that $\mathcal{O}(\mathfrak{g}, S, J)$ is a *triangular block of length m*.

The following theorem classifies the representation type of all triangular infinitesimal blocks. For its proof, see [FNP, Props. 5.3, 6.2, 7.1, 7.2]

Theorem 1. *Suppose $\mathcal{O}(\mathfrak{g}, S, J)$ is triangular of length m.*

(i) *If $m = 1$, then $\mathcal{O}(\mathfrak{g}, S, J)$ is semisimple.*
(ii) *If $m = 2$ or $m = 3$, then $\mathcal{O}(\mathfrak{g}, S, J)$ has finite representation type.*
(iii) *If $m = 4$, then $\mathcal{O}(\mathfrak{g}, S, J)$ has tame representation type.*
(iv) *If $m \geqslant 5$, then $\mathcal{O}(\mathfrak{g}, S, J)$ has wild representation type.*

4.2. Finite representation type

Suppose a linkage class of $\mathcal{O}(\mathfrak{g}, S, J)$ has m simple modules. If these simple modules are labeled $L_1, \dots, L_m$ and if the corresponding PVM's $V_1, \dots, V_m$ have radical filtration layers

$$\begin{array}{cccccc} V_1 & V_2 & V_3 & \cdots & V_{m-1} & V_m \\ \hline L_1 & L_2 & L_3 & \cdots & L_{m-1} & L_m \\ & L_1 & L_2 & \cdots & L_{m-2} & L_{m-1} \end{array} \tag{4.2.1}$$

then we say that the linkage class of $\mathcal{O}(\mathfrak{g}, S, J)$ is *uniserial of length* 2.

The following theorem says that there are very strict conditions placed on the structures of PVM's in a block having finite representation type.

Theorem 2. *$\mathcal{O}(\mathfrak{g}, S, J)$ has finite representation type if and only if the linkage classes of $\mathcal{O}(\mathfrak{g}, S, J)$ are uniserial of length* 2 *or triangular of length* 3.

For details, see [DoRe, Sec. 1] and [BN, Sec. 3.1].

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
Y_1 & & Y_2 \\
 & Z \arrow[ul] \arrow[ur] \arrow[d] & \\
 & W &
\end{tikzcd}$$

Fig. 2. A kite.

$$\begin{tikzcd}
 & L_4 \arrow[dl, shift left] \arrow[dr, shift left] & \\
L_2 \arrow[ur, shift left] \arrow[dr, shift left] & & L_3 \arrow[ul, shift left] \arrow[dl, shift left] \\
 & L_1 \arrow[ul, shift left] \arrow[ur, shift left] &
\end{tikzcd}$$

Fig. 3. The 'diamond' Ext^1-quiver.

4.3. *Wild representation type*

4.3.1. *Kite in the* Ext^1*-quiver*

Let Λ be a finite-dimensional 2-nilpotent algebra. Gabriel's Theorem [Gab1] asserts that the Ext^1-quiver of Λ separates into a union of quivers whose underlying graphs are Dynkin diagrams if and only if Λ has finite representation type. Furthermore, Dlab and Ringel [DlRi] proved that Λ has tame representation type if and only if the Ext^1-quiver of Λ separates into a union of quivers whose underlying graphs are Dynkin or extended Dynkin diagrams with at least one extended Dynkin diagram.

Suppose the infinitesimal block $\mathcal{O}_S^\mu$ is equivalent to the module category of a quasi-hereditary algebra A. Consider the finite-dimensional 2-nilpotent algebra $\Lambda = A/\operatorname{rad}^2 A$. Since $A \twoheadrightarrow \Lambda$, if Λ has wild representation type then so does A. Furthermore, Λ and A have the same Ext^1-quivers since each extension between simple modules arises as an extension between layers 0 and 1 of the radical filtration of some PVM in $\mathcal{O}_S^\mu$.

Now suppose the Ext^1-quiver of Λ contains a 'kite' with any orientation on the arrows, such as the kite shown in Fig. 2. Since the underlying graph is not a Dynkin diagram nor an extended Dynkin diagram, Λ must have wild representation type. This proves the following proposition.

Proposition 3. *If the* Ext^1*-quiver associated to* $\mathcal{O}_S^\mu$ *contains a kite, then* $\mathcal{O}_S^\mu$ *has wild representation type.*

4.3.2. *Diamond infinitesimal blocks*

The following argument is an adaptation of the argument in [FNP, Sec. 4.2] proving that $\mathcal{O}(A_1 \times A_1, \emptyset, \emptyset)$ has wild representation type. Suppose $\mathfrak{g}$ is any simple Lie algebra and $\mathcal{O}_S^\mu$ has exactly four simple modules L_1, L_2, L_3, L_4 and the corresponding PVM's have radical filtration layers:

$$\begin{array}{cccc}
V_1 & V_2 & V_3 & V_4 \\
\hline
L_1 & \begin{array}{c} L_2 \\ L_1 \end{array} & \begin{array}{c} L_3 \\ L_1 \end{array} & \begin{array}{c} L_4 \\ L_2 \quad L_3 \\ L_1 \end{array}
\end{array} \tag{4.3.1}$$

By noting the simple modules in layers 0 and 1 of each PVM, we have the Ext^1-quiver for $\mathcal{O}_{\mathcal{S}}$ depicted in Fig. 3. Using reciprocity, we can compute the radical filtration layers of the projective indecompos-

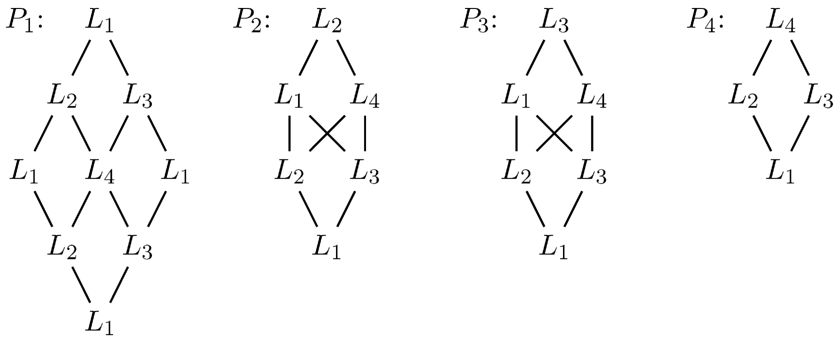

Fig. 4. Projective indecomposable modules for ‘diamond’.

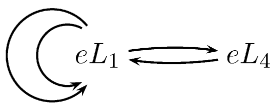

Fig. 5. Subquiver of eAe.

able modules [Irv]. These are shown in Fig. 4, where a line between simple modules represents an extension between them.

Let $P = P_1 \oplus P_2 \oplus P_3 \oplus P_4$ and set $A = \mathrm{End}_{\mathcal{O}_S^\mu}(P)^{\mathrm{op}}$, so $\mathcal{O}_S^\mu$ is Morita equivalent to the category of finitely generated A-modules. Consider the idempotent $e = 1_{L_1} + 1_{L_4}$. Now, A has wild representation type whenever the algebra eAe has wild representation type [Erd, I.4.7], and so localizing at e and using the structure of P_1 and P_4, we conclude that the quiver of eAe has a subquiver shown in Fig. 5. [Erd, I.10.8(i)] implies that eAe has wild representation type and therefore $\mathcal{O}_S^\mu$ has wild representation type.

One application of diamonds and kites is the following proposition due to [BN]. We say that four distinct elements $w, x_1, x_2, y \in W$ form a *diamond* if $y < x_i < w$ for $i = 1, 2$ and $l(w) = l(y) + 2$.

Proposition 4 *(Boe–Nakano). If ${}^S W^J$ contains a diamond, then $\mathcal{O}(\mathfrak{g}, S, J)$ has wild representation type.*

This follows because a diamond in ${}^S W^J$ gives rise to either a kite in the Ext^1-quiver or else a linkage class of $\mathcal{O}(\mathfrak{g}, S, J)$ which has exactly four simple modules with corresponding PVM's with structures as in (4.3.1).

5. Representation type of infinitesimal blocks in type F_4 and G_2

We determine the representation type of each infinitesimal block $\mathcal{O}(\Phi, S, J)$ when Φ is of type F_4 or G_2 using the results in Sections 3 and 4. Most of the results were found using a computer. This was done by first generating ${}^S W^J$ to see if it is non-empty. If it contains at least four elements, we generate the Hasse diagram of ${}^S W^J$ induced by the Bruhat order on W, and look for diamonds. If no diamonds are found or there are only two or three elements in ${}^S W^J$, we generate the radical filtrations of the PVM's in the infinitesimal block using the U_α-algorithm and look for kites, uniserial linkage classes, triangular linkage classes, or singleton linkage classes.

First, we adopt the following notation for the subroot systems of F_4 and G_2. If $T \subseteq \Delta$ contains only long roots (i.e., $T \subseteq \{\alpha_1, \alpha_2\}$ in F_4 or $T = \{\alpha_2\}$ in G_2), then Φ_T is denoted A_1 or A_2; if T contains only short roots (i.e., $T \subseteq \{\alpha_3, \alpha_4\}$ in F_4 or $T = \{\alpha_1\}$ in G_2), then Φ_T is denoted $\widetilde{A}_1$ or $\widetilde{A}_2$. Since there are two long simple roots and two short simple roots in type F_4, if $T = \{\alpha_2\}$, then we will denote Φ_T by A_1' and similarly if $T = \{\alpha_4\}$, then we will denote Φ_T by $\widetilde{A}_1'$.

Table 1
Representation type of $\mathcal{O}(G_2, \Phi_S, \Phi_J)$.

		Φ_J			
		G_2	A_1	$\widetilde{A}_1$	$\emptyset$
Φ_S	G_2	–	–	–	SS (1)
	A_1	–	SS (2)	SS (3)	F (6)
	$\widetilde{A}_1$	–	SS (3)	SS (2)	F (6)
	$\emptyset$	SS (1)	W^T (6)	W^T (6)	W^D (12)

The representation types of the infinitesimal blocks for types G_2 and F_4 are tabulated in Tables 1 and 2, respectively. The representation type of the infinitesimal block $\mathcal{O}(\Phi, S, J)$ is given on the row labeled with the root system Φ_S and the column labeled by the root system Φ_J. SS means the block is semisimple, F means it has finite representation type, and W means it has wild representation type; there are no tame blocks for these algebras. A dash $(-)$ means $\mathcal{O}(\mathfrak{g}, S, J) = 0$. The number below each SS in the table represents the number of simple modules in the semisimple block; this is also the number of linkage classes in the block. The number(s) below each F and W indicate the number of simple modules in the corresponding block; pairs of numbers indicate that the infinitesimal block splits into two linkage classes having the specified numbers of simple modules. For example, note that $\mathcal{O}(F_4, A_2, \widetilde{A_2})$ has two linkage classes, and they have, respectively, 20 and 12 simple modules for a total of 32 simple modules in the infinitesimal block. If an infinitesimal block is not semisimple, then the block does not split into more than two linkage classes in types F_4 and G_2.

If Φ is of type F_4 or G_2, all the infinitesimal blocks of finite representation type are composed of linkage classes which are uniserial length two as in (4.2.1). The superscript above each W indicates what condition was used to determine that the infinitesimal block has wild representation type. If there is a diamond in the poset of ${}^SW^J$, then it is marked W^D in the table. If the poset contains no diamonds but there is a kite in the Ext^1-quiver, then it is marked W^K in the table. If the infinitesimal block is triangular of length at least five, then it is marked with W^T. Notice that as one moves right in a row or down in a column, one expects to eventually find diamonds in the poset of ${}^SW^J$.

Partition the subroot systems of G_2 as

$$[G_2] := \{G_2\}, \qquad [A_1] := \{A_1, \widetilde{A}_1\}, \qquad [\emptyset] := \{\emptyset\}$$

and partition the subroot systems of F_4 as

$$[F_4] := \{F_4\}, \qquad [B_3] := \{B_3\}, \qquad [C_3] := \{C_3\},$$
$$[B_2] := \{A_2 \times \widetilde{A}'_1, B_2, \widetilde{A}_2 \times A_1\}, \qquad [A_2] := \{A_2\},$$
$$[\widetilde{A}_2] := \{\widetilde{A}_2\}, \qquad [A_1 \times \widetilde{A}_1] := \{A_1 \times \widetilde{A}_1, A_1 \times \widetilde{A}'_1, A'_1 \times \widetilde{A}'_1\},$$
$$[A_1] := \{A_1, A'_1, \widetilde{A}_1, \widetilde{A}'_1\}, \qquad [\emptyset] := \{\emptyset\}.$$

These equivalence classes are enclosed by horizontal and vertical lines in Tables 1 and 2.

Using the data in Tables 1 and 2, one can make some observations. To facilitate what follows, for the remainder of the paper the statement "representation type of $\mathcal{O}(\Phi, S, J)$" will mean one of the four mutually exclusive conditions for the block: zero, semisimple, finite representation type (but not semisimple), or wild representation type.

First, observe that the horizontal and vertical lines in Tables 1 and 2 divide the tables into rectangles with entries in each rectangle having the same representation type. This proves the following.

Table 2
Representation type of $\mathcal{O}(F_4, \Phi_S, \Phi_J)$.

Φ_S \ Φ_J	F_4	B_3	C_3	$A_2 \times \widetilde{A}'_1$	$\widetilde{A}_2 \times A_1$	B_2	A_2	$\widetilde{A}_2$	$A_1 \times \widetilde{A}_1$	$A'_1 \times \widetilde{A}'_1$	$A_1 \times \widetilde{A}'_1$	A_1	A'_1	$\widetilde{A}_1$	$\widetilde{A}'_1$	$\emptyset$
F_4	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	SS (1)
B_3	–	–	–	–	–	–	–	SS (1)	F (2)	F (2)	F (2)	W^K (6)	W^K (6)	W^K (9)	W^K (9)	W^D (24)
C_3	–	–	–	–	–	–	SS (1)	–	F (2)	F (2)	F (2)	W^K (9)	W^K (9)	W^K (6)	W^K (6)	W^D (24)
$A_2 \times \widetilde{A}'_1$	–	–	–	SS (3)	SS (5)	SS (4)	F (6)	F (6,6)	W^K (17)	W^K (17)	W^K (17)	W^D (36)	W^D (36)	W^D (44)	W^D (44)	W^D (96)
$\widetilde{A}_2 \times A_1$	–	–	–	SS (5)	SS (3)	SS (4)	F (6,6)	F (6)	W^K (17)	W^K (17)	W^K (17)	W^D (44)	W^D (44)	W^D (36)	W^D (36)	W^D (96)
B_2	–	–	–	SS (4)	SS (4)	SS (9)	F (6,6)	F (6,6)	W^K (24)	W^K (24)	W^K (24)	W^D (60)	W^D (60)	W^D (60)	W^D (60)	W^D (144)
A_2	–	–	SS (1)	W^T (6)	W^T (6,6)	W^T (6,6)	W^D (12)	W^D (20,12)	W^D (36)	W^D (36)	W^D (36)	W^D (72)	W^D (72)	W^D (48,48)	W^D (96)	W^D (192)
$\widetilde{A}_2$	–	SS (1)	–	W^T (6,6)	W^T (6)	W^T (6,6)	W^D (20,12)	W^D (12)	W^D (36)	W^D (36)	W^D (36)	W^D (96)	W^D (48,48)	W^D (72)	W^D (72)	W^D (192)
$A_1 \times \widetilde{A}_1$	–	F (2)	F (2)	W^K (17)	W^K (17)	W^K (24)	W^D (36)	W^D (36)	W^D (61)	W^D (61)	W^D (61)	W^D (132)	W^D (132)	W^D (132)	W^D (132)	W^D (288)
$A'_1 \times \widetilde{A}'_1$	–	F (2)	F (2)	W^K (17)	W^K (17)	W^K (24)	W^D (36)	W^D (36)	W^D (61)	W^D (61)	W^D (61)	W^D (132)	W^D (132)	W^D (132)	W^D (132)	W^D (288)
$A_1 \times \widetilde{A}'_1$	–	F (2)	F (2)	W^K (17)	W^K (17)	W^K (24)	W^D (36)	W^D (36)	W^D (61)	W^D (61)	W^D (61)	W^D (132)	W^D (132)	W^D (132)	W^D (132)	W^D (288)
A_1	–	W^K (6)	W^K (9)	W^D (36)	W^D (44)	W^D (60)	W^D (72)	W^D (96)	W^D (132)	W^D (132)	W^D (132)	W^D (264)	W^D (264)	W^D (288)	W^D (288)	W^D (576)
A'_1	–	W^K (6)	W^K (9)	W^D (36)	W^D (44)	W^D (60)	W^D (72)	W^D (96)	W^D (132)	W^D (132)	W^D (132)	W^D (264)	W^D (264)	W^D (288)	W^D (288)	W^D (576)
$\widetilde{A}_1$	–	W^K (9)	W^K (6)	W^D (44)	W^D (36)	W^D (60)	W^D (96)	W^D (72)	W^D (132)	W^D (132)	W^D (132)	W^D (288)	W^D (288)	W^D (264)	W^D (264)	W^D (576)
$\widetilde{A}'_1$	–	W^K (9)	W^K (6)	W^D (44)	W^D (36)	W^D (60)	W^D (96)	W^D (72)	W^D (132)	W^D (132)	W^D (132)	W^D (288)	W^D (288)	W^D (264)	W^D (264)	W^D (576)
$\emptyset$	SS (1)	W^D (24)	W^D (24)	W^D (96)	W^D (96)	W^D (144)	W^D (192)	W^D (192)	W^D (288)	W^D (288)	W^D (288)	W^D (576)	W^D (576)	W^D (576)	W^D (576)	W^D (1152)

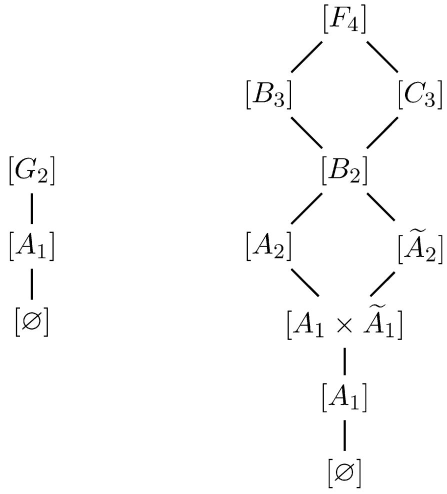

Fig. 6. Partial ordering on equivalence classes of subroot systems of G_2 (left) and F_4 (right).

Observation 1. If Φ is of type F_4 or G_2, then $\mathcal{O}(\Phi, S, J)$ and $\mathcal{O}(\Phi, S', J')$ have the same representation type whenever Φ_S and $\Phi_{S'}$ belong to the same equivalence class and Φ_J and $\Phi_{J'}$ belong to the same equivalence class.

We will define the representation type of $\mathcal{O}(\Phi, [\Phi_S], [\Phi_J])$ to be the representation type of $\mathcal{O}(\Phi, S, J)$; this is well defined because of Observation 1.

Observation 2. If Φ is of type F_4 or G_2, then $\mathcal{O}(\Phi, [\Phi_S], [\Phi_J])$ has the same representation type as $\mathcal{O}(\Phi, [\Phi_J], [\Phi_S])$ except in the following cases: each of

$$\mathcal{O}\bigl(G_2, [A_1], [\emptyset]\bigr), \quad \mathcal{O}\bigl(F_4, [B_2], [A_2]\bigr), \quad \mathcal{O}\bigl(F_4, [B_2], [\widetilde{A}_2]\bigr)$$

has finite representation type, but

$$\mathcal{O}\bigl(G_2, [\emptyset], [A_1]\bigr), \quad \mathcal{O}\bigl(F_4, [A_2], [B_2]\bigr), \quad \mathcal{O}\bigl(F_4, [\widetilde{A}_2], [B_2]\bigr)$$

have wild representation type. In particular, $\mathcal{O}(\Phi, [\Phi_S], [\Phi_J])$ is not zero (resp., semisimple) if and only if $\mathcal{O}(\Phi, [\Phi_J], [\Phi_S])$ is not zero (resp., semisimple).

One observes this by noting that Tables 1 and 2 are symmetric across the main diagonal except in the cases mentioned.

Now define a partial ordering on the equivalence classes of subroot systems of G_2 (resp., F_4) by the left (resp., right) Hasse diagram in Fig. 6. Denote both of these partial orderings by $\prec$ (or by $\preccurlyeq$ to include the possibility of equality). We are now ready for the main theorem.

Theorem 5. *Let Φ be of type F_4 or G_2.*

(i) *Suppose $S \subseteq \Delta$.*
 (a) *There exists an equivalence class $\overline{[\Phi_S]}$ such that if $\overline{[\Phi_S]} \prec [\Phi_J]$ then $\mathcal{O}(\Phi, [\Phi_S], [\Phi_J]) = 0$. Furthermore, if $\mathcal{O}(\Phi, [\Phi_S], [\Phi_J]) = 0$, then $\overline{[\Phi_S]} \prec [\Phi_J]$ or $[\Phi_J]$ and $\overline{[\Phi_S]}$ are incomparable. In particular, $\mathcal{O}(\Phi, [\Phi_S], \overline{[\Phi_S]}) \neq 0$.*

Table 3
Associated equivalence classes for G_2.

$[\Phi_T]$	$\overline{[\Phi_T]}$ or $\overline{\overline{[\Phi_T]}}$	$\underline{[\Phi_T]}$	$\underline{\underline{[\Phi_T]}}$
$[G_2]$	$[\emptyset]$	$[\emptyset]$	$[\emptyset]$
$[A_1]$	$[A_1]$	$[\emptyset]$	$[A_1]$
$[\emptyset]$	$[G_2]$	$[G_2]$	$[A_1]$

(b) *There exists an equivalence class $\underline{[\Phi_S]}$ such that $[\Phi_J] \prec \underline{[\Phi_S]}$ if and only if $\mathcal{O}(\Phi, [\Phi_S], [\Phi_J])$ has wild representation type.*
(c) *If $\underline{[\Phi_S]} \prec \overline{[\Phi_S]}$, then $\mathcal{O}(\Phi, [\Phi_S], \overline{[\Phi_S]})$ is semisimple and $\mathcal{O}(\Phi, [\Phi_S], \underline{[\Phi_S]})$ has finite representation type.*
(d) *If $\overline{[\Phi_S]} \prec \underline{[\Phi_S]}$, then for all $J \subseteq \Delta$, $\mathcal{O}(\Phi, [\Phi_S], [\Phi_J])$ is either zero or has wild representation type. In particular $\mathcal{O}(\Phi, [\Phi_S], \overline{[\Phi_S]})$ has wild representation type and $\mathcal{O}(\Phi, [\Phi_S], \underline{[\Phi_S]}) = 0$.*
(e) *If $\overline{[\Phi_S]} = \underline{[\Phi_S]}$, then $\mathcal{O}(\Phi, [\Phi_S], \overline{[\Phi_S]})$ is semisimple or has finite representation type.*
(f) *If $\overline{[\Phi_S]}$ is incomparable to $\underline{[\Phi_S]}$, then $[\Phi_S] \in \{[A_2], [\widetilde{A}_2], [A_1 \times \widetilde{A}_1]\}$ in which case $\overline{[\Phi_S]}, \underline{[\Phi_S]} \in \{[B_3], [C_3]\}$ and neither $\mathcal{O}(\Phi, [\Phi_S], \overline{[\Phi_S]})$ nor $\mathcal{O}(\Phi, [\Phi_S], \underline{[\Phi_S]})$ has wild representation type.*

(ii) *Suppose $J \subseteq \Delta$.*
(a) *There exists an equivalence class $\overline{\overline{[\Phi_J]}}$ such that if $\overline{\overline{[\Phi_J]}} \prec [\Phi_S]$ then $\mathcal{O}(\Phi, [\Phi_S], [\Phi_J]) = 0$. Furthermore, if $\mathcal{O}(\Phi, [\Phi_S], [\Phi_J]) = 0$, then $\overline{\overline{[\Phi_J]}} \prec [\Phi_S]$ or $[\Phi_S]$ and $\overline{\overline{[\Phi_J]}}$ are incomparable. In particular, $\mathcal{O}(\Phi, \overline{\overline{[\Phi_J]}}, [\Phi_J]) \neq 0$.*
(b) *There exists an equivalence class $\underline{\underline{[\Phi_J]}}$ such that $[\Phi_S] \prec \underline{\underline{[\Phi_J]}}$ if and only if $\mathcal{O}(\Phi, [\Phi_S], [\Phi_J])$ has wild representation type.*
(c) *If $\underline{\underline{[\Phi_J]}} \prec \overline{\overline{[\Phi_J]}}$, then $\mathcal{O}(\Phi, \overline{\overline{[\Phi_J]}}, [\Phi_J])$ is semisimple and $\mathcal{O}(\Phi, \underline{\underline{[\Phi_J]}}, [\Phi_J])$ has finite representation type.*
(d) *If $\overline{\overline{[\Phi_J]}} \prec \underline{\underline{[\Phi_J]}}$, then for all $S \subseteq \Delta$, $\mathcal{O}(\Phi, [\Phi_S], [\Phi_J])$ is either zero or has wild representation type. In particular $\mathcal{O}(\Phi, \overline{\overline{[\Phi_J]}}, [\Phi_J])$ has wild representation type and $\mathcal{O}(\Phi, \underline{\underline{[\Phi_J]}}, [\Phi_J]) = 0$.*
(e) *If $\overline{\overline{[\Phi_J]}} = \underline{\underline{[\Phi_J]}}$, then $\mathcal{O}(\Phi, \overline{\overline{[\Phi_J]}}, [\Phi_J])$ is semisimple or has finite representation type.*
(f) *If $\overline{\overline{[\Phi_J]}}$ is incomparable to $\underline{\underline{[\Phi_J]}}$, then $[\Phi_J] = [A_1 \times A_1]$ and $\overline{\overline{[\Phi_J]}}, \underline{\underline{[\Phi_J]}} \in \{[B_3], [C_3]\}$ and both $\mathcal{O}(\Phi, [\Phi_S], \overline{\overline{[\Phi_J]}})$ and $\mathcal{O}(\Phi, [\Phi_S], \underline{\underline{[\Phi_J]}})$ have finite representation type.*

Proof. If $T \subseteq \Delta$, then the (not in general unique) equivalence classes $\overline{[\Phi_T]}$, $\underline{[\Phi_T]}$, $\overline{\overline{[\Phi_T]}}$, and $\underline{\underline{[\Phi_T]}}$ corresponding to a given $[\Phi_T]$ are listed in Table 3 for G_2 and Table 4 for F_4. The proof of the theorem now follows by inspecting Tables 1 and 2. □

The somewhat complicated description of the representation type of infinitesimal blocks for a Lie algebra of type F_4 as given in Theorem 5 arises in part because for an arbitrary equivalence class $[\Phi_T]$, the equivalence classes $\overline{[\Phi_T]}$, $\underline{[\Phi_T]}$, $\overline{\overline{[\Phi_T]}}$, and $\underline{\underline{[\Phi_T]}}$ are not in general unique. If one introduces three more "virtual classes" that act as placeholders in the partial ordering, one gets the top-to-bottom symmetric extended Hasse diagram given in Fig. 7. (These virtual classes come from Richardson orbits which are not labeled by standard root systems.)

For each equivalence class $[\Phi_T]$ of a subroot system of F_4, let $[\Phi_T]^*$ be the (possibly virtual) class obtained by reflecting $[\Phi_T]$ across the horizontal line of symmetry in the extended Hasse diagram of F_4. Now, given an equivalence class $[\Phi_T]$, define the (possibly virtual) class $\widehat{[\Phi_T]}$ by

$$\widehat{[\Phi_T]} = \begin{cases} [\Phi_T]^* & \text{if } [\Phi_T] \neq [B_2], \\ [B_2] & \text{if } [\Phi_T] = [B_2] \end{cases}$$

Table 4
Associated equivalence classes for F_4.

$[\Phi_T]$	$\overline{[\Phi_T]}$ or $\overline{\overline{[\Phi_T]}}$	$\underline{[\Phi_T]}$	$\underline{\underline{[\Phi_T]}}$
$[F_4]$	$[\emptyset]$	$[\emptyset]$	$[\emptyset]$
$[B_3]$	$[\widetilde{A}_2]$	$[A_1 \times \widetilde{A}_1]$	$[A_1 \times \widetilde{A}_1]$
$[C_3]$	$[A_2]$	$[A_1 \times \widetilde{A}_1]$	$[A_1 \times \widetilde{A}_1]$
$[B_2]$	$[B_2]$	$[A_2]$, $[\widetilde{A}_2]$	$[B_2]$
$[A_2]$	$[C_3]$	$[B_3]$, $[C_3]$	$[B_2]$
$[\widetilde{A}_2]$	$[B_3]$	$[B_3]$, $[C_3]$	$[B_2]$
$[A_1 \times \widetilde{A}_1]$	$[B_3]$, $[C_3]$	$[B_3]$, $[C_3]$	$[B_3]$, $[C_3]$
$[A_1]$	$[B_3]$, $[C_3]$	$[F_4]$	$[F_4]$
$[\emptyset]$	$[F_4]$	$[F_4]$	$[F_4]$

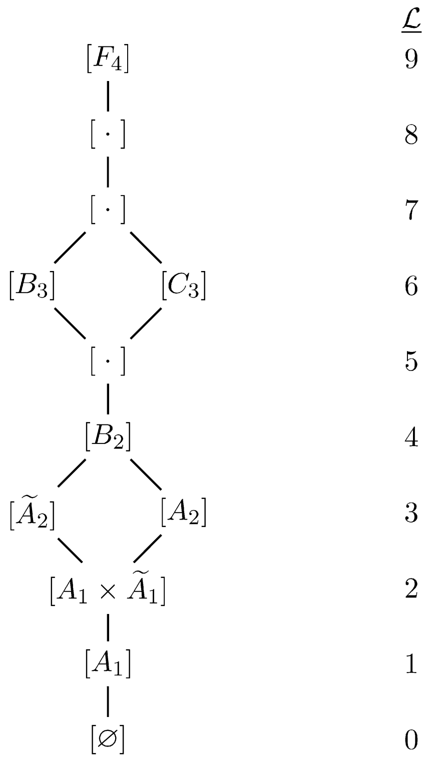

Fig. 7. Extended poset of equivalence classes in F_4.

and let $\mathcal{L}(\mathcal{C}) \in \{0, 1, \ldots, 9\}$ denote the layer index of the (possibly virtual) class $\mathcal{C}$ in the extended Hasse diagram in Fig. 7.

Theorem 6. *Let Φ be of type F_4 and let $[\Phi_S]$, $[\Phi_J]$ be equivalence classes of (true) subroot systems of F_4.*

(i) *The following are equivalent.*
  (a) *$\mathcal{O}(F_4, [\Phi_S], [\Phi_J])$ is non-zero.*
  (b) *$[\Phi_J] \preccurlyeq \widehat{[\Phi_S]}$.*
  (c) *$[\Phi_S] \preccurlyeq \widehat{[\Phi_J]}$.*

(ii) *The following are equivalent.*

(a) $\mathcal{O}(F_4, [\Phi_S], [\Phi_J])$ *is semisimple.*
(b) $[\Phi_J] = \widehat{[\Phi_S]}$.
(c) $[\Phi_S] = \widehat{[\Phi_J]}$.

(iii) *The following are equivalent.*
(a) $\mathcal{O}(F_4, [\Phi_S], [\Phi_J])$ *has finite representation type.*
(b) $\mathcal{L}([\Phi_J]) = \mathcal{L}(\widehat{[\Phi_S]}) - 1$.
(c) $[\Phi_J] = [A_2]$ *and* $[\Phi_J] = [B_2]$; *or* $[\Phi_J] = [\widetilde{A}_2]$ *and* $[\Phi_S] = [B_2]$; *or* $[\Phi_J] \neq [A_2], [\widetilde{A}_2], [B_2]$ *and* $\mathcal{L}([\Phi_S]) = \mathcal{L}(\widehat{[\Phi_J]}) - 1$.

(iv) *The following are equivalent.*
(a) $\mathcal{O}(F_4, [\Phi_S], [\Phi_J])$ *has wild representation type.*
(b) $\mathcal{L}([\Phi_J]) \leqslant \mathcal{L}(\overline{[\Phi_S]}) - 2$.
(c) $[\Phi_J] = [A_2]$ *or* $[\Phi_J] = [\widetilde{A}_2]$ *and* $\mathcal{L}([\Phi_S]) \leqslant \mathcal{L}(\widehat{[\Phi_J]}) - 3$; *or* $[\Phi_J] \neq [A_2], [\widetilde{A}_2]$ *and* $\mathcal{L}([\Phi_S]) \leqslant \mathcal{L}([\Phi_J]^*) - 2$.

Proof. Note that Observation 2 implies that interchanging $[\Phi_S]$ and $[\Phi_J]$ does not affect whether the block is 0 or semisimple. One can see in Table 2 that $\mathcal{O}(F_4, [\Phi_S], [\Phi_J])$ is semisimple exactly when $\overline{[\Phi_S]} = [\Phi_J]$ and zero exactly when $[\Phi_J] \preccurlyeq \widehat{[\Phi_S]}$ to obtain (i) and (ii). For each $[\Phi_S]$, if $[\Phi_J]$ is a true equivalence class of a subroot system which lies exactly one layer down from $\overline{[\Phi_S]}$, then one verifies in Table 2 that $\mathcal{O}(F_4, [\Phi_S], [\Phi_J])$ has finite representation type and that this exhausts all of the infinitesimal blocks with finite type. Furthermore, according to Observation 2, the only cases for which $\mathcal{O}(F_4, [\Phi_S], [\Phi_J])$ has finite representation type but $\mathcal{O}(F_4, [\Phi_J], [\Phi_S])$ does not have finite representation type are when $[\Phi_S] = [B_2]$ and $[\Phi_J] = [A_2]$ or $[\Phi_J] = [\widetilde{A}_2]$ in which case $\mathcal{O}(F_4, [\Phi_J], [\Phi_S])$ has wild representation type. This proves (iii) and since we exhausted all cases but that of wild representation type, it also proves (iv). $\square$

In principle, one can use a computer to determine the representation types of the blocks in types E_6, E_7, and E_8. However, their sizes make this a much more difficult task; therefore, a better approach would be desirable in these cases. Preliminary calculations suggest that there is likely an analogue of Theorem 6 for all the exceptional Lie algebras.

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